

# NON-ARCHIMEDEAN GEOMETRY OF ARTIN FANS

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**ABSTRACT.** The purpose of this article is to study the role of Artin fans in tropical and non-Archimedean geometry. Artin fans are logarithmic algebraic stacks that are logarithmically étale over the base field  $k$ . Despite their seemingly abstract definition the geometry of Artin fans can be described completely in terms of combinatorial objects, so called Kato stacks, a stack-theoretic generalization of K. Kato's notion of a fan. Every logarithmic scheme admits a tautological strict morphism  $\phi_X : X \rightarrow \mathcal{A}_X$  to an associated Artin fan. The main result of this article is that, on the level of underlying topological spaces, the natural functorial tropicalization map of  $X$  is nothing but the non-Archimedean analytic map associated to  $\phi_X$  by applying Thuillier's generic fiber functor. As an immediate consequence we obtain that both the tropicalization and the skeleton of a logarithmic scheme canonically carry the structure of a non-Archimedean analytic stack.

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## 1. INTRODUCTION

**1.1. Kato fans and tropicalization.** Let  $k$  be an algebraically closed field that is endowed with the trivial absolute value. Inspired by the construction of the non-Archimedean skeleton associated to a toroidal embedding in [Thu07], in [Uli13] the author has defined a *tropicalization map*  $\text{trop}_X : X^\natural \rightarrow \bar{\Sigma}_X$  associated to every fine and saturated logarithmic scheme locally of finite type over  $k$  that generalizes earlier constructions for split algebraic tori (see [EKL06], [Gub07], and [Gub13]) and toric varieties (see [Kaj08] and [Pay09]).

The heuristic that guides this construction is based on the fact that in the special case that  $X$  has *no monodromy*, there is natural strict *characteristic morphism*  $\bar{\phi}_X : (X, \mathcal{O}_X/\mathcal{O}_X^*) \rightarrow F_X$  into the *Kato fan*  $F_X$  associated to  $X$ , a locally monoidal space that captures the combinatorial type of the

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logarithmic strata of  $X$  (see [Kat94]). The tropicalization map  $\text{trop}_X$  is simply the "analytification" of this characteristic morphism in a suitable topological sense.

The advantage and the beauty of working with Kato fans mostly stems from their dual nature as algebraic objects, whose geometry is completely determined by combinatorics. Their usage, however, suffers from two major inconveniences:

- (1) Not every fine and saturated logarithmic scheme admits a characteristic morphism into a Kato fan. In fact, as soon as the logarithmic structure on  $X$  encodes *monodromy*, a Kato fan does not exist (see [Uli13] Example 4.12).
- (2) Kato fans live in the category of sharp monoidal spaces. So, despite their inherently algebraic nature, most of the techniques of algebraic geometry cannot directly be applied to Kato fans.

**1.2. Artin fans and Kato stacks.** Both problems can be resolved by working with *Artin fans*, a notion that has originally been introduced in [AW13] and [ACMW14] in the context of logarithmic Gromov-Witten theory, but can be implicitly traced back to the work of Olsson [Ols03] on classifying stacks of logarithmic structures.

*Artin fans* are logarithmic algebraic stacks that are logarithmically étale over the base field  $k$ . Despite their seemingly abstract definition the geometry of Artin fans can be described completely in terms of combinatorics, since every Artin fan is étale locally isomorphic to a stack quotient  $[X/T]$ , where  $X$  is a  $T$ -toric variety. In fact, we show in Theorem 4.1 below that the 2-category of Artin fans (with faithful monodromy) is equivalent to the category of *Kato stacks* (with faithful monodromy), i.e. to a category of geometric stacks over the category of Kato fans (also see [ACM<sup>+</sup>15, Section 5.2]). This description suggests that Artin fans in addition to the local toric stack structure also encode a second kind of stackyness, the one coming from the monodromy of the logarithmic structure.

Given a fine and saturated logarithmic scheme  $X$ , there always is an Artin fan  $\mathcal{A}_X$  (with faithful monodromy) as well as a natural strict morphism  $\phi_X : X \rightarrow \mathcal{A}_X$ , which one can think of as a generalization of the characteristic morphism from [Uli13], lifted to the category of logarithmic algebraic stacks.

**1.3. Analytic geometry of Artin fans.** Expanding on the heuristics that the tropicalization map is the "analytification" of the characteristic morphism, it seems natural to expect that, after applying Thuillier's [Thu07] generic fiber functor  $(.)^\natural$  (see Section 5 below), the non-Archimedean analytic map  $\phi_X^\natural : X^\natural \rightarrow \mathcal{A}_X^\natural$  should, at least topologically, agree with  $\text{trop}_X$ . The following Theorem 1.1 makes this expectation precise.

**Theorem 1.1.** *Let  $X$  be a fine and saturated logarithmic scheme locally of finite type over  $k$ . There is a natural homeomorphism  $\mu_X : |\mathcal{A}_X^\natural| \xrightarrow{\sim} \bar{\Sigma}_X$  that makes the diagram*

$$\begin{array}{ccc} & X^\natural & \\ \phi_X^\natural \swarrow & & \searrow \text{trop}_X \\ |\mathcal{A}_X^\natural| & \xrightarrow[\mu_X]{\sim} & \bar{\Sigma}_X \end{array}$$

*commute.*

Suppose now that  $X$  is logarithmically smooth over  $k$ . By [Uli13, Theorem 1.2] and [Thu07] the tropicalization map  $\text{trop}_X$  admits a continuous section  $J_X : \bar{\Sigma}_X \rightarrow X^\triangleright$  such that the composition  $p_X = J_X \circ \text{trop}_X$  is a strong deformation retraction of  $X^\triangleright$  onto the *skeleton*  $\mathfrak{S}(X)$  associated to  $X$ . Therefore Theorem 1.1 immediately implies the following Corollary 1.2.

**Corollary 1.2.** *Let  $X$  be a logarithmically smooth scheme locally of finite type over  $k$ . Then there is a natural homeomorphism  $\tilde{\mu}_X : |\mathcal{A}_X^\triangleright| \xrightarrow{\sim} \mathfrak{S}(X)$  that makes the diagram*

$$\begin{array}{ccc} & X^\triangleright & \\ \phi_X^\triangleright \swarrow & & \searrow p_X \\ |\mathcal{A}_X^\triangleright| & \xrightarrow[\tilde{\mu}_X]{\sim} & \mathfrak{S}(X) \end{array}$$

*commute.*

Theorem 1.1 and Corollary 1.2 in particular show that both the generalized extended complex and the skeleton of a logarithmic stack canonically carry the structure of an analytic stack. The author expects that this is only the starting point of a longer program of endowing the different kinds of non-Archimedean skeletons (see for example [Ber90], [Ber99], [BPR11], [GRW14], and [MN15]) with the structure of an analytic stack and thereby categorifying these a priori only topological constructions within the realm of analytic geometry.

**1.4. The case of toric varieties.** Let  $T$  be a split algebraic torus with character lattice  $M$  and cocharacter lattice  $N$ . Let  $X = X(\Delta)$  be a  $T$ -toric variety defined by a rational polyhedral fan. In [Kaj08] and [Pay09] the authors construct a partial compactification  $N_{\mathbb{R}}(\Delta)$  of  $N_{\mathbb{R}} = N \otimes \mathbb{R}$  that allows them to define a natural continuous tropicalization map  $\text{trop}_\Delta : X^{\text{an}} \rightarrow N_{\mathbb{R}}(\Delta)$ . For a  $T$ -invariant open subset  $U_\sigma = \text{Spec } k[S_\sigma]$  corresponding to a cone  $\sigma$  in  $\Delta$  we have  $N_{\mathbb{R}}(\sigma) = \text{Hom}(S_\sigma, \overline{\mathbb{R}})$ , where  $\overline{\mathbb{R}} = \mathbb{R} \sqcup \{\infty\}$  with the natural additive monoid structure, and  $N_{\mathbb{R}}(\sigma)$  is endowed with the topology of pointwise convergence. On  $U_\sigma$  the tropicalization map

$$\text{trop}_\sigma : U_\sigma^{\text{an}} \longrightarrow N_{\mathbb{R}}(\sigma)$$

is given by sending  $x \in U_\sigma^{\text{an}}$  to the homomorphism  $s \mapsto -\log |x^s|_x$ .

By [Uli13, Proposition 7.1] we obtain the following Corollary 1.3.

**Corollary 1.3.** *Suppose that  $X$  is a complete  $T$ -toric variety. Then there is a natural homeomorphism  $\mu_X : |\mathcal{A}_X^\triangleright| \xrightarrow{\sim} N_{\mathbb{R}}(\Delta)$  that makes the diagram*

$$\begin{array}{ccc} & X^{\text{an}} & \\ \phi_X^\triangleright \swarrow & & \searrow \text{trop}_\Delta \\ |\mathcal{A}_X^\triangleright| & \xrightarrow[\mu_X]{\sim} & N_{\mathbb{R}}(\Delta) \end{array}$$

*commute.*

Since  $\mathcal{A}_X = [X/T]$  by Example 3.8 below, Corollary 1.3 suggests a new perspective on the tropicalization of toric varieties as the non-Archimedean analytic stack quotient

$$X^\triangleright \longrightarrow [X^\triangleright/T^\triangleright].$$

With some technical modifications Corollary 1.3 can be generalized to all toric varieties defined over any non-Archimedean field, not necessarily carrying the trivial absolute value (see [Uli14, Theorem 1.1]). As explained in the introduction of [Uli14], this adds a further layer to the analogy between the tropicalization map in the non-Archimedean and the moment map in the Archimedean world.

**1.5. Applications and further developments.** Artin fans have been introduced in [AW13] and [ACMW14] in order to study the toroidal birational geometry of the moduli space of logarithmically stable maps. The main point hereby is that they form a convenient and flexible framework to study toroidal modifications via subdivisions, expanding on K. Kato's construction via his theory of fans in [Kat94, Section 10] as well as the original approach in [KKMSD73, Section II.2]. In the upcoming [Uli16] the author intends to use the framework of Artin fans and its connections to toroidal modifications in order to further develop his logarithmic reinterpretation of Tevelev's theory of tropical compactifications (see [Tev07] and [Uli15]).

Theorem 1.1 and its toric generalization in [Uli14] have already found applications to tropical geometry. In [Ran15b] Ranganathan shows that, while not every tropical curve  $\Gamma \subseteq \mathbb{R}^n$  is realizable by a curve in a  $T \simeq \mathbb{G}^n$ -toric variety  $X$ , it is possible to always find a curve over the Artin fan  $\mathcal{A}_X = [X/T]$  that tropicalizes to  $\Gamma$ . The main reason why this approach works can be traced back to the vanishing of certain cohomological logarithmic obstruction groups over  $\mathcal{A}_X$  (see [CFPU14, Section 3]). Furthermore, in [Ran15a] Theorem 1.1 forms a crucial technical ingredient towards Ranganathan's conceptual explanation of the classical correspondence theorem between rational curves on toric varieties and their tropical counterparts (see [Mik05] and [NS06]) in terms of logarithmic Gromov-Witten theory and via the tropical geometry of moduli spaces (see [ACP15]).

Theorem 4.1 below suggests that the framework of Kato stacks (as developed in Section 2) might be a good candidate to describe the stack-theoretic properties of the moduli space of tropical curves. In particular, one can hope to make sense of the notion of a universal curve over the tropical moduli space in analogy with the classic theory of Knudsen [Knu83] for the moduli space of algebraic curves. This is an ongoing project by Melody Chan, Renzo Cavalieri, Jonathan Wise, and the author.

In [Lor15, Section 13] Lorscheid provides an enrichment of the structure of the Kato fan  $F_X$  of a logarithmic scheme  $X$  without monodromy within the category of ordered blue schemes. The advantage of his approach is that now one can also endow the tropicalization of a closed subscheme of  $X$  with an algebraic structure. The author expects that there is a general stack-theoretic object, an "ordered blue stack", which generalizes both ordered blue schemes and Kato stacks, and which categorifies the tropical geometry of subspaces of logarithmic schemes with monodromy (or, more generally, of logarithmic stacks).

Finally, it would of course be desirable to have a relative theory of Artin fans over arbitrary valuation rings of rank one. The idea in this case would be to have Artin fans that are étale locally isomorphic to stack quotients of toric schemes (see [KKMSD73, Section IV.3] and [Gub13, Section 6 and 7]) by their big tori. We refrain from including this here, since the necessary technical foundations for logarithmic structures over arbitrary valuation rings of rank one and their connections to tropical and non-Archimedean geometry are still work-in-progress.

**1.6. Conventions.** A *monoid* is a commutative semigroup with a unit element and will be mostly written additively. Throughout this article we assume that all monoids are fine and saturated (fs) and implicitly amalgamated sums will be taken in the category of fs monoids.

The letter  $k$  will always stand for an algebraically closed field  $k$  that is endowed with the trivial absolute value and the trivial logarithmic structure  $k^* \subseteq k$ . We are working in the category  $\mathbf{LSch}$  of fine and saturated logarithmic schemes  $X = (\underline{X}, M_X)$  in the sense of [Kat89], which are locally of finite type over  $k$ , unless mentioned otherwise. In particular, we shall simply refer to a fine and saturated logarithmic scheme that is locally of finite type over  $k$  as a *logarithmic scheme*.

Recall that a morphism  $f : X \rightarrow X'$  of logarithmic schemes is said to be *strict*, if the canonical morphism  $f^*M_{X'} \rightarrow M_X$  is an isomorphism. The category  $\mathbf{LSch}$  is endowed with the *strict étale topology*, the coarsest topology that makes the functor  $X \mapsto \underline{X}$  continuous with respect to the étale topology.

The term *logarithmic stack* will always refer to a fine and saturated logarithmic stack that is locally of finite type over  $k$ , i.e. an algebraic stack  $\underline{\mathcal{X}}$  locally of finite type over  $k$  that is endowed with a logarithmic structure  $M_{\mathcal{X}}$  in the lisse-étale topology. To every logarithmic stack  $\mathcal{X}$  we may associate a geometric stack over the  $\mathbf{LSch}_{\text{str.ét.}}$ , which (in a slight abuse of notation) we are also going to denote by  $\mathcal{X}$  (see [ACG<sup>+</sup>13, Section 4] for a short discussion of this issue).

Fix a logarithmic scheme  $S$ . In [Ols03] Olsson introduces the category  $\underline{\text{LOG}}_S$ , whose objects are morphisms  $T \rightarrow S$  of logarithmic schemes and whose arrows are strict morphisms over  $S$ . The association  $T \mapsto \underline{T}$  makes  $\underline{\text{LOG}}_S$  into a category fibered in groupoids over the category of  $\underline{S}$ -schemes. By [Ols03, Theorem 1.1]  $\underline{\text{LOG}}_S$  is an algebraic stack that is locally of finite presentation over  $S$ . The stack  $\underline{\text{LOG}}_S$  is referred to as the *classifying stack of logarithmic structures over  $S$* .

Given a logarithmic scheme  $X$  over  $S$ , there is a natural *tautological morphism*  $\underline{X} \rightarrow \underline{\text{LOG}}_S$  determined by associating to  $f \in \underline{X}(T)$  the logarithmic scheme  $(T, f^*M_X)$  over  $S$ . A morphism  $T \rightarrow T'$  over  $\underline{X}$  hereby naturally induces a strict morphism  $(T, f^*M_X) \rightarrow (T', (f')^*M_X)$  by the universal property of pullbacks. The stack  $\underline{\text{LOG}}_S$  can be endowed with a universal logarithmic structure such that for every logarithmic scheme  $X$  over  $S$  the tautological morphism  $X \rightarrow \text{LOG}_S$  is strict.

Let  $\underline{X}$  be a scheme over  $\underline{S}$ . In this situation, the data of a logarithmic structure on  $\underline{X}$  over  $S$  is equivalent to the data of a tautological morphism  $\underline{X} \rightarrow \underline{\text{LOG}}_S$ . It has been shown in [Ols03, Section 4] that a logarithmic morphism  $X \rightarrow S$  is logarithmically étale (or logarithmically smooth) if and only if the morphism  $\underline{X} \rightarrow \underline{\text{LOG}}_S$  is étale (or smooth respectively).

More generally, every logarithmic algebraic stack  $\mathcal{X}$  over  $S$  admits a strict tautological morphism  $\mathcal{X} \rightarrow \text{LOG}_S$  that fully determines the logarithmic structure on  $\underline{\mathcal{X}}$ . Using lisse-étale charts one can deduce that a logarithmic morphism  $\mathcal{X} \rightarrow S$  is logarithmically étale (or logarithmically smooth) if and only if the tautological morphism  $\underline{\mathcal{X}} \rightarrow \underline{\text{LOG}}_S$  is étale (or smooth respectively).

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## 2. KATO STACKS

Denote the dual category of the category of sharp monoids by  $\mathbf{Aff}$ . We refer to an object of this category as an *affine Kato fan*. As proposed in [Kat94] (also see [Uli13, Section 3.1]) one

can visualize the affine Kato fan corresponding to a sharp monoid  $P$  as a sharp monoidal space  $\text{Spec } P$ , the *spectrum* of  $P$ :

- As a set  $\text{Spec } P$  is the set of prime ideals of  $P$ , i.e. the set of subsets  $\mathfrak{p}$  of  $P$  such that  $\mathfrak{p} \cdot \mathfrak{p} \subseteq \mathfrak{p}$  for all  $\mathfrak{p} \in \text{Spec } P$  and the complement  $P - \mathfrak{p}$  is a monoid;
- the topology on  $\text{Spec } P$  is generated by the basic open subsets  $D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$  for  $f \in P$ ; and
- the structure sheaf is defined by associating to  $D(f)$  the sharp localization  $\overline{P}_f = P_f/P_f^*$  of  $P$  along  $f$ .

The category **Aff** carries a Grothendieck topology, the *big Zariski topology*; its covers are generated by open subsets coming from sharp localizations  $P \rightarrow \overline{P}_f$  along submonoids generated by an element  $f \in P$ .

**Lemma 2.1.** *The big Zariski topology and the chaotic topology on **Aff** are equivalent.*

*Proof.* This is an immediate consequence of the fact that the only Zariski open subset of  $\text{Spec } P$  that contains the maximal ideal of  $P$  is  $D(0) = \text{Spec } P$  itself.  $\square$

**Definition 2.2.** A *Kato fan*  $F$  is a sharp monoidal space that admits a covering  $U_i$  by open subsets that are isomorphic to affine Kato fans  $\text{Spec } P_i$ .

A *morphism* of Kato fans is a morphism of sharp monoidal spaces. Throughout this article we assume that every Kato fan is *locally fine and saturated (fs)*, i.e. the monoids  $P_i$  in the covering of  $F$  above can be chosen to be fine and saturated. We refer the reader to [GM15b] and [Gil16] for the state of the art in the theory of Kato fans as well as to [Uli13, Section 3] for many explicit examples.

Every Kato fan defines a presheaf

$$\begin{aligned} h_F : \mathbf{Aff}^{\text{op}} &\longrightarrow \mathbf{Sets} \\ P &\longmapsto F(P) = \text{Hom}(\text{Spec } P, F) \end{aligned}$$

and by the Yoneda Lemma the association  $F \mapsto h_F$  defines a fully faithful functor from the category of Kato fans to the category of presheaves on **Aff**.

**Proposition 2.3.** *For a Kato fan  $F$  the presheaf  $h_F$  is a sheaf.*

*Proof.* By Lemma 2.1 we only have to check the sheaf axioms for the chaotic topology and here the sheaf axioms are automatically fulfilled.  $\square$

In the following we are going to identify a Kato fan  $F$  with its functor of points  $h_F$  and simply write  $F$  for both the locally monoidal space and the functor.

**Proposition 2.4.** *The category of locally fine and saturated Kato fans admits fiber products.*

*Proof.* Given a diagram

$$\begin{array}{ccc} & G & \\ & \downarrow & \\ F & \longrightarrow & H \end{array}$$

of Kato fans, we have to construct a Kato fan  $F \times_H G$  together with morphism to  $F$  and  $G$  that makes the above diagram cartesian in the category of Kato fans. If  $F = \text{Spec } P$ ,  $G = \text{Spec } Q$ , and



$H = \operatorname{Spec} S$  are all affine, we can set  $F \times_H G = \operatorname{Spec} \overline{P \oplus_S Q}$ , where  $P \oplus_S Q$  is the amalgamated sum in the category of fs monoids. This is a fiber product, since for every sharp monoidal space  $X$  we have:

$$\begin{aligned} \operatorname{Hom}(X, F \times_H G) &= \operatorname{Hom}(P \oplus_S Q, \mathcal{O}_X(X)) \\ &= \operatorname{Hom}(P, \mathcal{O}_X(X)) \times_{\operatorname{Hom}(S, \mathcal{O}_X(X))} \operatorname{Hom}(Q, \mathcal{O}_X(X)) . \end{aligned}$$

The general case now follows by glueing.  $\square$

**Definition 2.5.** A morphism  $f : F \rightarrow G$  of locally fs Kato fans is said to be *strict*, if the induced map  $f^* \mathcal{O}_G \rightarrow \mathcal{O}_F$  is an isomorphism

**Proposition 2.6.** (i) A morphism  $f : F \rightarrow G$  between two Kato fans is strict if and only if it is a local isomorphism.

(ii) The class of strict morphisms is stable under composition and base change.

*Proof.* A morphism  $f : \operatorname{Spec} Q \rightarrow \operatorname{Spec} P$  is strict if and only if its the induced map  $P \rightarrow Q$  is an isomorphism. This shows (i), which, in turn, immediately implies (ii).  $\square$

The category **Aff** can be endowed with a further Grothendieck topology, the *strict topology*, whose coverings are given by smooth morphisms. By Proposition 2.6 (ii) the strict topology is, in fact, equal to the big Zariski topology on **Aff**.

**Definition 2.7.** A *Kato stack* is a stack  $\mathcal{F}$  over the site **Aff** fulfilling the following two properties:

- (i) The diagonal morphism  $\mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$  is representable by Kato fans.
- (ii) There is a morphism  $U \rightarrow \mathcal{F}$  from a Kato fan  $U$  to  $\mathcal{F}$  that is strict and surjective.

We are going to refer to the strict and surjective morphism  $U \rightarrow \mathcal{F}$  and, in a slight abuse of notation, to  $U$  itself as an *atlas* of  $\mathcal{F}$ . Note hereby that  $U \rightarrow \mathcal{F}$  is representable, since the diagonal of  $\mathcal{F}$  is representable. The category of Kato stacks (denoted by **K.St.**) is the full subcategory of categories fibered in groupoids over **Aff**, whose objects are Kato stacks. Note hereby that a category fibered in groupoids over **Aff**<sub>chaotic</sub> is already automatically a stack, since descent with respect to the chaotic topology is always trivial.

Throughout this article we are again going to assume that all of our Kato stacks are locally fine and saturated, i.e. there is an atlas  $U \rightarrow \mathcal{F}$  such that  $U$  is locally fine and saturated.

**Example 2.8.** By Proposition 2.3 every Kato fan  $F$  defines a Kato stack, which, in a slight abuse of notation, we are simply going to denote by  $F$  as well.

**Proposition 2.9.** Let  $(R \rightrightarrows U)$  be a strict surjective groupoid object in the category of Kato fans. Then the quotient stack  $[U/R]$  is a Kato stack.

The proof of Proposition 2.9 proceeds in complete analogy with the algebraic case (see [Sta16, Tag 04TJ]); due to the triviality of descent with respect to the chaotic topology, we may, however, avoid some technicalities.

*Proof of Proposition 2.9.* The groupoid  $(R \rightrightarrows U)$  defines a functor

$$\mathbf{Aff}^{\text{op}} \longrightarrow \mathbf{Groupoids}$$

by associating to  $T \in \mathbf{Aff}$  the groupoid  $(R(T) \rightrightarrows U(T))$  and the associated category  $[U/R]$  fibered in groupoids is already a stack, since descent with respect to the chaotic topology always holds.

In order to show that the diagonal of  $[U/R]$  is representable by Kato fans we only have to show that for an affine Kato fan  $T$  and two objects  $x, y \in [U/R](T)$  the sheaf  $\text{Isom}(x|_T, y|_T)$  is representable by a Kato fan. But this follows, since we have a natural cartesian diagram

$$\begin{array}{ccc} \text{Isom}(x|_T, y|_T) & \longrightarrow & R \\ \downarrow & & \downarrow \\ T & \xrightarrow{(x|_T, y|_T)} & U \times U. \end{array}$$

The natural morphism  $R \rightarrow U \times_{[U/R]} U$  is an equivalence and therefore, for every  $x \in [U/R](T)$ , there are natural equivalences

$$U \times_{[U/R]} T \simeq (U \times_{[U/R]} U) \times_{s, U, x} T \simeq R \times_{s, U, x} T.$$

By assumption the projection morphism  $R \times_U T \rightarrow T$  is surjective and strict as a base change of  $s : R \rightarrow U$  and thus the natural map  $U \rightarrow [U/R]$  is a strict atlas of  $[U/R]$ .  $\square$

A *presentation* of a Kato stack  $\mathcal{F}$  consists of a groupoid  $(R \rightrightarrows U)$  in **Fans** together with an equivalence  $[U/R] \simeq \mathcal{F}$ . Let  $U \rightarrow \mathcal{F}$  be a strict atlas of a Kato stack  $\mathcal{F}$ . The fiber product  $U \times_{\mathcal{F}} U$  is representable by a Kato fan  $R$  and the projections  $\text{pr}_0, \text{pr}_1 : R \rightrightarrows U$ , together with the natural composition morphism

$$R \times_U R \simeq U \times_{\mathcal{F}} U \times_{\mathcal{F}} U \xrightarrow{\text{pr}_{02}} U \times_{\mathcal{F}} U \simeq R,$$

define a groupoid in the category of Kato fans. Its quotient stack  $[U/R]$  is equivalent to the Kato stack  $\mathcal{F}$  we started with. We refer the reader to [Sta16, Tag 04T3] for an analogous construction in the category of schemes.

**Example 2.10.** Let  $G$  be a finite group acting on an affine Kato fan  $U = \text{Spec } P$  by automorphisms. For every  $G$ -invariant open affine subset  $V$  of  $U$  we write  $G_V$  for the image of  $G$  in  $\text{Aut}(V)$ . We define the *twisted product*  $R_G = U * G$  of  $U$  and  $G$  as the colimit of the diagram consisting of the products  $V \times G_V$  for all  $G$ -invariant open affine subsets of  $U$  and morphisms

$$V' \times \{g'\} \xrightarrow{\subseteq} V \times \{g\},$$

whenever  $V'$  is a  $G$ -invariant open affine subset of  $V$  and  $g$  is in the preimage of  $g'$  under the natural quotient map  $G_V \twoheadrightarrow G_{V'}$ .

The group operation on  $U$  induces a strict surjective groupoid  $R_G \rightrightarrows U$  in the category of Kato fans and, by Proposition 2.9, gives rise to a quotient stack

$$[U // G] = [U/R_G],$$

the *twisted group quotient* of  $U$  by  $G$ .

**Example 2.11** (The classifying stack FAN). Consider the trivial categorical fibration  $\mathbf{Aff} \rightarrow \mathbf{Aff}$  and denote by FAN the associated category fibered in groupoids, i.e. its restriction to the class of cartesian morphisms. For every affine Kato fan  $U = \text{Spec } P$  there is a canonical morphism  $U \rightarrow \text{FAN}$  that is unique up to equivalence. Moreover, we have

$$U \times_{\text{FAN}} U \simeq U * \text{Aut}(U)$$

and therefore FAN is a Kato stack that is locally isomorphic to the twisted group quotients  $[U // \text{Aut}(U)]$  for all affine Kato fans  $U$ .



**Definition 2.12.** A Kato stack  $\mathcal{F}$  is said to have *faithful monodromy*, if the tautological morphism  $\mathcal{F} \rightarrow \text{FAN}$  is representable.

For example, the twisted group quotients  $[\mathcal{U} // G]$  from Example 2.10 above are Kato stacks with faithful monodromy.

**Example 2.13.** Let  $G$  be a finite non-trivial group. By Proposition 2.9 the classifying stack

$$\text{BG} = [\text{pt}/G]$$

is a Kato stack, yet it does not have faithful monodromy.

**Remarks 2.14.** (i) Of course the definition of Kato stacks in this section is a special instance of the notion of geometric (1-)stack in the geometric context of Kato fans. We refer the avid reader to [Sim96], [TV08, Chapter 1.3], and [PY14, Section 2] for the general theory.

(ii) It would be very interesting to investigate whether the various forms of stacky fans, which arise as combinatorial models in the theory of toric stacks (see e.g. [BCS05], [FMN10], [GS15], and [GM15a]) can be realized as Kato stacks (without faithful monodromy!).

### 3. ARTIN FANS

**Definition 3.1.** An *Artin fan* is a logarithmic algebraic stack that is logarithmically étale over  $k$ .

In other words, an Artin fan is a logarithmic algebraic stack  $\mathcal{A}$  whose tautological morphism  $\mathcal{A} \rightarrow \text{LOG}_k$  is étale. If  $\mathcal{A} \rightarrow \text{LOG}_k$  is also representable, we say that  $\mathcal{A}$  has *faithful monodromy*. So an Artin fan with faithful monodromy is nothing but an étale sheaf over  $\text{LOG}_k$ .

**Example 3.2** (Artin cones). Let  $P$  be a monoid and write  $U_P$  for the affine toric variety  $\text{Spec } k[P]$  with big torus  $T = \text{Spec } k[P^{\text{gp}}]$ . As explained in [Ols03, Section 5] the quotient stack  $\mathcal{A}_P = [U_P/T]$  naturally carries a logarithmic structure making the morphism  $\mathcal{A}_P \rightarrow \text{LOG}_k$  representable and étale. Therefore  $\mathcal{A}_P$  is an Artin fan with faithful monodromy. We are going to refer to an Artin fan of the form  $\mathcal{A}_P$  as an *Artin cone*.

**Example 3.3** (Artin fans without monodromy). A homomorphism  $P \rightarrow Q$  of monoids induces a torus-invariant morphism  $U_Q \rightarrow U_P$  and therefore a representable logarithmic morphism  $\mathcal{A}_Q \rightarrow \mathcal{A}_P$  on the level of quotient stacks. If  $P \rightarrow Q$  is a face map, then  $\mathcal{A}_Q \rightarrow \mathcal{A}_P$  is a strict open immersion. Therefore we can associate to every Kato fan  $F$  an Artin fan

$$\mathcal{A}_F = \varinjlim \mathcal{A}_P$$

by glueing the  $\mathcal{A}_P$  over all open affine subsets  $U = \text{Spec } P$  of  $F$  such that there is a natural isomorphism  $(|\mathcal{A}_F|, \overline{M}_{\mathcal{A}_P}) \simeq F$  of sharp monoidal spaces. Artin fans of the form  $\mathcal{A}_F$  are said to be *without monodromy*.

The category of Artin fans (with faithful monodromy) is the full 2-subcategory of the 2-category of logarithmic algebraic stacks whose objects are Artin fans (with faithful monodromy) and will be denoted by **AF** (or **AF<sub>f.m.</sub>**, respectively).

The following Proposition 3.4 is a rephrasing of [ACMW14, Proposition 3.1.1].

**Proposition 3.4** ([ACMW14] Proposition 3.1.1). *Let  $X$  be a logarithmic scheme. Then there is an Artin fan  $\mathcal{A}_X$  with faithful monodromy as well as a strict morphism  $X \rightarrow \mathcal{A}_X$  that is initial among all strict morphisms into Artin fans with faithful monodromy.*

In other words, there is an Artin fan  $\mathcal{A}_X$  with faithful monodromy as well as a strict morphism  $X \rightarrow \mathcal{A}_X$  such that, if there is another strict morphism  $X \rightarrow \mathcal{A}$  into an Artin fan  $\mathcal{A}$  with faithful monodromy, then there is a unique strict morphism  $\mathcal{A}_X \rightarrow \mathcal{A}$  that makes the diagram

$$\begin{array}{ccc} & & \mathcal{A}_P \\ & \nearrow \phi_X & \downarrow \\ X & & \mathcal{A} \\ & \searrow & \end{array}$$

commute.

**Corollary 3.5.** *A strict morphism  $f : X \rightarrow Y$  of logarithmic schemes induces a strict morphism  $f^{\mathcal{A}} : \mathcal{A}_X \rightarrow \mathcal{A}_Y$  that makes the diagram*

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{A}_X \\ f \downarrow & & \downarrow f^{\mathcal{A}} \\ Y & \longrightarrow & \mathcal{A}_Y \end{array}$$

commute.

*Proof.* This immediately follows from the universal property of  $\phi_X : X \rightarrow \mathcal{A}_X$  applied to the strict composition  $\phi_X \circ f : X \rightarrow \mathcal{A}_Y$ .  $\square$

**Definition 3.6.** A Zariski logarithmic scheme  $X$  is said to be *small* if the locus of points  $x \in X$ , for which the restriction homomorphism

$$\Gamma(X, \overline{M}_X) \longrightarrow \overline{M}_{X,x}$$

is a bijection, is non-empty and irreducible.

Given a small logarithmic scheme  $X$  we write  $P_X$  for the sharp monoid  $\Gamma(X, \overline{M}_X)$ .

*Proof of Proposition 3.4.* For a logarithmic scheme  $X$  and a fine and saturated monoid  $P$ , there is a natural bijection

$$\mathrm{Hom}(P, \Gamma(X, \overline{M}_X)) \simeq \mathrm{Hom}(X, \mathcal{A}_P)$$

by [Ols03, Proposition 5.17], under which strict morphisms  $X \rightarrow \mathcal{A}_P$  on the right correspond precisely to homomorphisms  $P \rightarrow \overline{M}(X)$  on the left that lift to a chart of  $X$ . So, if  $X$  is a small Zariski logarithmic scheme, we have an Artin cone  $\mathcal{A}_X = \mathcal{A}_{P_X}$  and obtain a natural strict morphism

$$\phi_X : X \longrightarrow \mathcal{A}_X$$

that satisfies the above universal property by the following Lemma (see [ACMW14, Proposition 3.1.1] for details).

**Lemma 3.7.** *Let  $\mathcal{A}$  be an Artin fan with faithful monodromy and suppose there is a point  $x : \mathrm{Spec} k \rightarrow \mathcal{A}$  with  $\overline{M}_{\mathcal{A},x} = P$ . Then there is a unique strict morphism  $\mathcal{A}_P \rightarrow \mathcal{A}$  that makes the diagram*

$$\begin{array}{ccc} (\mathrm{Spec} k, P) & \xrightarrow{x} & \mathcal{A} \\ \downarrow & \nearrow & \downarrow \\ \mathcal{A}_P & \longrightarrow & \mathrm{LOG}_k \end{array}$$

commute. The left vertical arrow hereby refers to the strict embedding of the closed point into  $\mathcal{A}_P$ .

Let us now finish the proof of Proposition 3.4 and consider an arbitrary logarithmic scheme  $X$ . Choose a strict étale presentation  $R \rightrightarrows U$  of  $X$  by disjoint unions of small Zariski logarithmic schemes, i.e.  $U = \bigsqcup_i U_i$  and  $R = \bigsqcup_j R_j$  such that all  $U_i$  and  $R_j$  are small. In this case  $\mathcal{A}_U = \bigsqcup_i \mathcal{A}_{U_i}$  and  $\mathcal{A}_R = \bigsqcup_j \mathcal{A}_{R_i}$  fulfill the above universal property. We can now define  $\mathcal{A}_X$  as the colimit of the strict étale morphisms  $\mathcal{A}_R \rightrightarrows \mathcal{A}_U$  in the category of étale sheaves over  $\text{LOG}_k$  and the universal property of colimits implies that  $X \rightarrow \mathcal{A}_X$  has the desired universal property.  $\square$

*Proof of Lemma 3.7.* In order to prove Lemma 3.7 we may apply Proposition 3.4 in the special case of a logarithmically smooth scheme, since this is a stack-theoretic special case of the construction in [LMB00, Section (6.8)] (see [AW13, Section 2.1 and 2.2] for details). Use Kato's criterion for logarithmic smoothness to choose a strict, smooth, quasi-compact chart  $U \rightarrow \mathcal{A}$  around  $x$  from a small logarithmically smooth scheme  $U$  with  $P_U = P$ . Then the universal property of  $U \rightarrow \mathcal{A}_P$  gives rise to a unique strict morphism  $\mathcal{A}_P \rightarrow \mathcal{A}$  that makes the above diagram commute.  $\square$

**Example 3.8** (Toric varieties). Let  $P$  be a monoid. The Artin fan of the affine toric variety  $U_P = \text{Spec } k[P]$  is the Artin cone  $\mathcal{A}_P = [U_P/T]$  and the tautological morphism is the quotient map  $U_P \rightarrow \mathcal{A}_P$ . For a general  $T$ -toric variety  $Z$ , glueing over  $T$ -invariant open affine subsets yields that the associated Artin fan  $\mathcal{A}_Z$  is the quotient stack  $[Z/T]$  and the tautological morphism is the quotient map  $Z \rightarrow [Z/T]$ .

**Example 3.9** (Logarithmic schemes without monodromy). Recall now from [Uli13, Section 4.3] that a Zariski logarithmic scheme  $X$  is said to have *no monodromy* if there is a strict morphism  $(X, \overline{M}_X) \rightarrow F$  of sharp monoidal spaces into a Kato fan  $F$ . In this case, by [Uli13, Proposition 4.14], there is a strict morphism  $\overline{\phi}_X : (X, \overline{M}_X) \rightarrow F_X$  into a Kato fan  $F_X$  that is initial among all strict morphisms to Kato fans; the morphism  $\overline{\phi}_X : (X, \overline{M}_X) \rightarrow F_X$  is referred to as the *characteristic morphism* associated to  $X$ .

Moreover, the induced diagram

$$\begin{array}{ccc} & (X, \overline{M}_X) & \\ \downarrow |\phi_X| & & \searrow \overline{\phi}_X \\ (|\mathcal{A}_X|, \overline{M}_{\mathcal{A}_X}) & \xrightarrow{\sim} & F_X \end{array}$$

commutes: For, if  $X$  is small with respect to a point  $x \in X$ , we may set  $P = \overline{M}_{X,x}$  and both morphisms  $\phi_X : X \rightarrow \mathcal{A}_P$  and  $\overline{\phi}_X : (X, \overline{M}_X) \rightarrow \text{Spec } P$  are induced from a lift of  $P \rightarrow \overline{\mathcal{O}}_X$  to a chart  $P \rightarrow \mathcal{O}_X$  of  $M_X$ . The general case follows by glueing.

**Remark 3.10.** Given a (not necessarily strict) morphism  $f : X \rightarrow Y$  of logarithmic schemes, there is still a morphism  $\mathcal{A}(f) : \mathcal{A}_X \rightarrow \mathcal{A}_Y$  and the association is functorial in  $f$ , but the diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{A}_X \\ f \downarrow & & \downarrow \mathcal{A}(f) \\ Y & \longrightarrow & \mathcal{A}_Y \end{array}$$

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is not always commutative (see [ACM<sup>+</sup>15, 5.4.1] for a counterexample). Nevertheless, in this situation, there is a natural commutative diagram

$$\begin{array}{ccc} \mathrm{LOG}_X & \longrightarrow & \mathcal{A}_X \\ \mathrm{LOG}(f) \downarrow & & \downarrow \mathcal{A}(f) \\ \mathrm{LOG}_Y & \longrightarrow & \mathcal{A}_Y \end{array}$$

and, since the open immersion  $X \rightarrow \mathrm{LOG}_X$  induces a homeomorphism  $|X^\natural| \simeq |\mathrm{LOG}_X^\natural|$  (in the notation introduced in Section 5 below) this is still consistent with the functoriality of the tropicalization map, i.e. [Uli13, Theorem 1.1].

#### 4. THE CATEGORY OF ARTIN FANS

Denote the category of small logarithmic schemes by **smLSch**; it is naturally endowed with the restriction of the strict étale topology on **LSch**. Every logarithmic scheme admits a strict étale covering by a disjoint union of small logarithmic schemes. Therefore the pullback of a logarithmic stack to **smLSch** defines an equivalence of 2-categories and so, from now on, we are implicitly going to assume that all logarithmic stacks are defined over **smLSch**.

Consider the natural functor

$$\begin{aligned} \mathbf{smLSch} &\longrightarrow \mathbf{Aff} \\ X &\longmapsto \mathrm{Spec} P_X . \end{aligned}$$

Since a strict surjective étale morphism  $U \rightarrow X$  of small logarithmic schemes already induces an isomorphism  $P_X \simeq P_U$ , this functor is continuous and it therefore induces a natural *characteristic morphism*

$$\chi : \mathbf{Aff}_{\mathrm{chaotic}} \longrightarrow \mathbf{smLSch}_{\mathrm{str.ét.}}$$

between the corresponding sites.

The following Theorem 4.1 shows that Artin fans are lifts of Kato stacks to the category of logarithmic algebraic stacks, as claimed in the introduction.

**Theorem 4.1.** *The pushforward  $\mathcal{F} \mapsto \mathcal{A}_{\mathcal{F}} = \chi_* \mathcal{F}$  along the characteristic morphism induces an equivalence*

$$\mathbf{KSt}_{\mathrm{f.m.}} \xrightarrow{\sim} \mathbf{AF}_{\mathrm{f.m.}}$$

of 2-categories, i.e.

- (i) *given a Kato stack  $\mathcal{F}$  with faithful monodromy, the pushforward  $\mathcal{A}_{\mathcal{F}} = \chi_* \mathcal{F}$  is an Artin fan with faithful monodromy;*
- (ii) *given an Artin fan  $\mathcal{A}$  with faithful monodromy, there is a Kato stack  $\mathcal{F}$  with faithful monodromy such that  $\mathcal{A} \simeq \mathcal{A}_{\mathcal{F}}$ ; and*
- (iii) *for two Kato stacks  $\mathcal{F}$  and  $\mathcal{G}$  with faithful monodromy the natural arrow*

$$\begin{aligned} \mathrm{Hom}(\mathcal{F}, \mathcal{G}) &\longrightarrow \mathrm{Hom}(\mathcal{A}_{\mathcal{F}}, \mathcal{A}_{\mathcal{G}}) \\ f &\longmapsto \chi_* f \end{aligned}$$

*between the functor categories defines an equivalence.*

**Remark 4.2.** In fact, it is expected that the pushforward along  $\chi$  even induces a stronger 2-equivalence

$$\mathbf{KSt} \xrightarrow{\sim} \mathbf{AF}$$

that restricts to the equivalence in Theorem 4.1. For example, given a finite group  $G$ , the characteristic morphism  $\chi$  induces a correspondence between the classifying stacks  $BG$  in **KSt** and **AF**.

The remainder of this section is devoted to the proof of Theorem 4.1. We begin by considering an affine Kato fan  $\text{Spec } P$ . Since every small logarithmic scheme  $X$  with  $P_X = P$  admits a strict tautological morphism  $X \rightarrow \mathcal{A}_X = \mathcal{A}_P$  that is adjoint to the identity, we see that  $\chi_* \text{Spec } P = \mathcal{A}_P$ .

**Lemma 4.3.** *For two sharp monoids  $P$  and  $Q$  the natural map*

$$\text{Hom}(P, Q) \longrightarrow \text{Hom}(\mathcal{A}_Q, \mathcal{A}_P)$$

*is a bijection.*

*Proof.* By [Ols03, Proposition 5.17] we have

$$\text{Hom}(U_Q, \mathcal{A}_P) = \text{Hom}(P, \Gamma(U_Q, \overline{M}_{U_Q})) = \text{Hom}(P, Q)$$

and the left hand side is equal to  $\text{Hom}(\mathcal{A}_Q, \mathcal{A}_P)$ , since every morphism  $U_Q \rightarrow \mathcal{A}_P$  is  $T_Q = \text{Spec } k[Q^{\text{gp}}]$ -invariant.  $\square$

Let us now consider Artin fans without monodromy.

**Lemma 4.4.** *The association  $F \mapsto \mathcal{A}_F$  defines a fully faithful functor from the category of Kato fans into the category of Artin fans.*

*Proof.* Let  $F$  be a Kato fan. The Artin fan  $\mathcal{A}_F$  is a final element in the category of logarithmic schemes without monodromy and Kato fan  $F_X = F$  and therefore we have  $\chi_* F = \mathcal{A}_F$ . Glueing over open immersions together with Lemma 4.3 yields the claim.  $\square$

**Lemma 4.5.** *Let  $F$  be a Kato fan. A strict étale cover  $\mathcal{A} \rightarrow \mathcal{A}_F$  by an Artin fan  $\mathcal{A}$  with faithful monodromy also has no monodromy.*

*Proof.* We need to show that there is a Kato fan  $G$  such that  $\mathcal{A} = \mathcal{A}_G$ . Hereby we may assume that  $F = \text{Spec } P$  is affine. By Lemma 3.7 for every point  $x$  in  $\mathcal{A}$  such that  $\overline{M}_{\mathcal{A}, x} = P$  the strict étale cover  $\mathcal{A} \rightarrow \mathcal{A}_F$  has a section such that the diagram

$$\begin{array}{ccc} (\text{Spec } k, P) & \xrightarrow{x} & \mathcal{A} \\ \downarrow & \nearrow & \downarrow \\ \mathcal{A}_P & \longrightarrow & \text{LOG}_k \end{array}$$

commutes. This shows that  $\mathcal{A}$  is isomorphic to  $\mathcal{A}_P$  in a Zariski open neighborhood of  $x$ .  $\square$

Let us now consider the general case. The following Lemma 4.6 gives an explicit description of the pushforward  $\mathcal{A}_{\mathcal{F}} = \chi_* \mathcal{F}$ .

**Lemma 4.6.** *Let  $\mathcal{F}$  be a Kato stack. As a category fibered in groupoids over **smLSch** the stack  $\mathcal{A}_{\mathcal{F}} = \chi_* \mathcal{F}$  is given by associating to  $T \in \mathbf{smLSch}$  with  $P_T = \Gamma(T, \overline{M}_T)$  the groupoid of pairs  $(\xi, \gamma)$  consisting of an element in  $\xi \in \mathcal{A}_{P_T}(T)$  and an element in  $\gamma \in \mathcal{F}(P_T) = \mathcal{F}(\text{Spec } P_T)$ .*

*Proof.* This is [Sta16, Tag 04WA] phrased in the terminology used in this article.  $\square$

By Lemma 4.6 every element in  $\mathcal{A}_{\mathcal{F}}(T)$  or, in other words, a morphism  $T \rightarrow \mathcal{A}_{\mathcal{F}}$  factors as  $T \rightarrow \mathcal{A}_{P_T} \rightarrow \mathcal{A}_{\mathcal{F}}$ , where the morphism  $T \rightarrow \mathcal{A}_{P_T}$  corresponds to  $\xi \in \mathcal{A}_{P_T}(T)$ .

**Example 4.7.** The logarithmic stack  $\text{LOG}_k$  is the Artin fan  $\mathcal{A}_{\text{FAN}}$  associated to the classifying stack FAN of all Kato fans.

**Lemma 4.8.** *Let  $f : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of Kato stacks. If  $f$  is representable, so is the induced morphism  $\mathcal{A}(f) : \mathcal{A}_{\mathcal{F}} \rightarrow \mathcal{A}_{\mathcal{G}}$ . Moreover, if  $f$  is also strict, then  $\mathcal{A}(f)$  is a strict étale morphism.*

*Proof.* Given a morphism  $U \rightarrow \mathcal{A}_{\mathcal{G}}$  from a small logarithmic scheme  $U$ , we need to show that the base change  $\mathcal{A}_{\mathcal{F}} \times_{\mathcal{A}_{\mathcal{G}}} U$  is representable by a logarithmic scheme. By Lemma 4.6 we may canonically factor the morphism  $U \rightarrow \mathcal{A}_{\mathcal{G}}$  as  $U \rightarrow \mathcal{A}_{P_U} \rightarrow \mathcal{A}_{\mathcal{G}}$  and obtain a tower of cartesian squares as follows:

$$\begin{array}{ccc} \mathcal{A}_{\mathcal{F}} \times_{\mathcal{A}_{\mathcal{G}}} U & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathcal{A}_{\mathcal{F}} \times_{\mathcal{A}_{\mathcal{G}}} \mathcal{A}_{P_U} & \longrightarrow & \mathcal{A}_{P_U} \\ \downarrow & & \downarrow \\ \mathcal{A}_{\mathcal{F}} & \xrightarrow{\mathcal{A}(f)} & \mathcal{A}_{\mathcal{G}}. \end{array}$$

Since  $f$  is representable by Kato fans, we have

$$\mathcal{A}_{\mathcal{F}} \times_{\mathcal{A}(\mathcal{G})} \mathcal{A}_{P_U} \simeq \mathcal{A}_{\mathcal{H}}$$

for some Kato fan  $\mathcal{H}$ . By Lemma 4.4 the morphism  $\mathcal{A}_{\mathcal{H}} \rightarrow \mathcal{A}_{P_U}$  is representable and this implies that  $\mathcal{A}_{\mathcal{F}} \times_{\mathcal{A}_{\mathcal{G}}} U$  is representable by a logarithmic scheme  $V$ .

For the second part, we need to show that, if  $f$  is strict, then  $V \rightarrow U$  is a strict étale morphism. Since  $f$  is a strict morphism, so is its base change  $\mathcal{H} \rightarrow \text{Spec } P_U$ . But then the induced map  $\mathcal{A}_{\mathcal{H}} \rightarrow \mathcal{A}_{P_U}$  is a local isomorphism and therefore, in particular, a strict étale morphism, which implies the claim.  $\square$

*Proof of Theorem 4.1.* In order to prove (i) we first need to show that  $\mathcal{A}_{\mathcal{F}} = \chi_* \mathcal{F}$  is a logarithmic algebraic stack. Lemma 4.8 immediately implies that the diagonal morphism of  $\mathcal{A}_{\mathcal{F}} \rightarrow \mathcal{A}_{\mathcal{F}} \times \mathcal{A}_{\mathcal{F}}$  is representable. In order to construct an atlas for  $\mathcal{A}_{\mathcal{F}}$ , we choose a strict surjective morphism  $U \rightarrow \mathcal{F}$  from a Kato fan  $U$  onto the Kato stack  $\mathcal{F}$ . By Lemma 4.8 the induced map  $\mathcal{A}_U \rightarrow \mathcal{A}_{\mathcal{F}}$  is a surjective strict étale morphism and its composition with a strict lisse-étale atlas of  $\mathcal{A}_U$  gives rise to a strict lisse-étale atlas of  $\mathcal{A}_{\mathcal{F}}$ .

In order to see that  $\mathcal{A}_{\mathcal{F}}$  is an Artin fan with faithful monodromy we again use Lemma 4.8 and therefore we need to verify that, for every Kato stack  $\mathcal{F}$  with faithful monodromy, the tautological morphism  $\mathcal{F} \rightarrow \text{FAN}$  is representable and strict. Let  $U = \text{Spec } P$  be an affine Kato fan; write  $G = \text{Aut}(U)$  and denote by  $N$  the normal closure in  $G$  of the group of automorphisms of  $U$  in  $\mathcal{F}$ . Then we have a natural isomorphism

$$\mathcal{F} \times_{\text{FAN}} U \simeq U * (G/N)$$

and this implies our claim.

For (ii) let  $\mathcal{A}$  be an Artin fan with faithful monodromy. We consider the stack  $\mathcal{F}_{\mathcal{A}}$  over  $\mathbf{Aff}$ , whose fiber over  $\text{Spec } P$  is the functor groupoid  $\text{Hom}(\mathcal{A}_P, \mathcal{A})$ . The diagonal of  $\mathcal{F}_{\mathcal{A}}$  is representable



by Lemma 4.5. In order to construct a strict atlas of  $\mathcal{F}_{\mathcal{A}}$  we consider an strict lisse-étale atlas

$$\bigsqcup \mathcal{U}_i \longrightarrow \mathcal{A}$$

consisting of a disjoint union of small logarithmic schemes  $\mathcal{U}_i$ . Write  $P_i = P_{\mathcal{U}_i} = \Gamma(\mathcal{U}_i, \overline{M})$ . Then Proposition 3.4 yields a strict surjective étale morphism  $\bigsqcup \mathcal{A}_{P_i} \rightarrow \mathcal{A}$  and, in turn, a strict surjective morphism  $\bigsqcup \text{Spec } P_i \rightarrow \mathcal{F}_{\mathcal{A}}$ . Using the explicit description of the pushforward in Lemma 4.6 above, we obtain that

$$\chi_* \mathcal{F}_{\mathcal{A}} \simeq \mathcal{A}$$

and the tautological morphism  $\mathcal{F}_{\mathcal{A}} \rightarrow \text{FAN}$  is representable, since  $\mathcal{A} \rightarrow \text{LOG}_k$  is.

Finally, Lemma 4.6 together with Lemma 4.3 immediately imply (iii).  $\square$

## 5. STACKY GENERIC FIBERS OVER TRIVIALY VALUED FIELDS

Let  $X$  be a scheme that is locally of finite type over  $k$ . Consider the functor that associates to an analytic space  $Y$  the set of morphism  $f : Y \rightarrow X$  of locally ringed spaces such that the following conditions hold:

- For every open affine  $U = \text{Spec } A$  in  $X$  and every affinoid domain  $V = \mathcal{M}(\mathcal{B})$  in  $Y$  with  $f(V) \subseteq U$  the induced homomorphism  $f^\# : A \rightarrow \mathcal{B}$  is continuous and bounded, i.e. we have  $|f^\#(a)| \leq 1$  for every  $a \in A$ .
- For every point  $y \in Y$  the induced homomorphism

$$\mathcal{O}_{X, \phi(y)} \longrightarrow \mathcal{O}_{Y, y} \longrightarrow \mathcal{H}(y)$$

has values in  $\mathcal{H}(y)^\circ$  and therefore induces a local homomorphism  $\mathcal{O}_{X, \phi(y)} \rightarrow \mathcal{H}(y)^\circ$ .

In [Thu07, Proposition et Définition 1.3] Thuillier shows that this functor is representable by an analytic space  $X^\triangleright$ . The association  $X \mapsto X^\triangleright$  defines a functor

$$(\cdot)^\triangleright : \mathbf{Schemes}_{\text{loc.f.t./}k} \longrightarrow \mathbf{An}_k$$

and there is a natural morphism from  $X^\triangleright$  to an analytic domain in the usual analytic space  $X^{\text{an}}$  associated to  $X$ . If  $X$  is separated over  $k$ , this morphism is injective, and, if  $X$  is proper over  $k$ , we have  $X^\triangleright = X^{\text{an}}$ .

**Example 5.1.** Let  $X = \text{Spec } A$  be an affine scheme of finite type over  $k$ . The analytic space  $X^{\text{an}}$  associated to  $X$  is the set of multiplicative seminorms on  $A$  that restrict to the trivial norm on  $k$  and  $X^\triangleright$  is the affinoid domain in  $X^{\text{an}}$  consisting of all seminorms that are bounded, i.e. that fulfill  $|a| \leq 1$  for all  $a \in A$ .

In this section we are going to develop the necessary theory to generalize the  $(\cdot)^\triangleright$ -functor to algebraic stacks that are locally of finite type over  $k$  and study its basic properties.

**5.1. Topology of G-analytic stacks.** Let  $k$  be any non-Archimedean field. Throughout this section we denote by  $\mathbf{An}_k$  the category of non-Archimedean analytic spaces in the sense of [Ber93] over a non-Archimedean field  $k$ . The category  $\mathbf{An}_k$  is endowed with the *étale topology* (see [Ber93, Section 4]); the resulting site will be denoted by  $(\mathbf{An}_k)_{\text{ét}}$ .

Recall that in [Uli14] we have already introduced a theory of geometric stacks in the category of analytic spaces, so called *analytic stacks*. An alternative approach to non-Archimedean analytic stacks can be found in [Yu14] and [PY14].

**Definition 5.2** ([Uli14] Section 2). Let  $k$  be a non-Archimedean field.

(i) An *étale analytic space* is a sheaf

$$X : (\mathbf{An}_k)_{\text{ét}} \longrightarrow \mathbf{Sets}$$

that admits a representable surjective étale morphism  $U \rightarrow X$  from an analytic space  $U$ .

(ii) A stack  $\mathcal{X}$  over the étale site  $(k - \mathbf{An})_{\text{ét}}$  is said to be *analytic*, if the diagonal morphism

$$\Delta : \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$$

is representable by étale analytic spaces, and there is a morphism  $U \rightarrow \mathcal{X}$  from an analytic space  $U$  that is surjective,  $G$ -smooth, and universally submersive.

In order to generalize the  $(.)^\triangleright$ -functor below to algebraic stacks we need to extend these notions as follows.

**Definition 5.3.** (i) A *G-étale analytic space*  $X$  is an analytic stack such that its underlying category is fibered in sets and there is an atlas  $U \rightarrow X$  that is  $G$ -étale, surjective, and universally submersive.

(ii) A stack  $\mathcal{X}$  over  $(k - \mathbf{An})_{\text{ét}}$  is said to be *G-analytic*, if the diagonal morphism

$$\Delta : \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$$

is representable by  $G$ -étale analytic spaces, and there is a morphism  $U \rightarrow \mathcal{X}$  from an analytic space  $U$  that is  $G$ -smooth, surjective, and universally submersive.

Note that, since the categories underlying a  $G$ -étale analytic space are fibered in sets, the category of  $G$ -étale analytic spaces is really only a 1-category. Therefore the above definition of  $G$ -analytic stacks does not leave the familiar realm of 2-categories.

The basic results concerning groupoid presentations and their quotients, as e.g. developed in [Uli14], immediately generalize to the framework of  $G$ -analytic stacks. In particular, given a  $G$ -smooth, surjective, and universally submersive groupoid  $(R \rightrightarrows U)$  in the category  $G$ -étale analytic spaces, we find that the quotient stack  $[U/R]$  is  $G$ -analytic.

**Definition 5.4.** Let  $\mathcal{X}$  be a  $G$ -analytic stack over  $k$ .

(i) A *point* of  $\mathcal{X}$  is a pair  $(K, \phi)$  consisting of a non-Archimedean extension  $K$  of  $k$  and a morphism  $\phi : \mathcal{M}(K) \rightarrow \mathcal{X}$ .

(ii) Two points  $(K, \phi)$  and  $(L, \psi)$  of  $\mathcal{X}$  are said to be *equivalent*, if there is a common non-Archimedean extension  $\Omega$  of both  $K$  and  $L$  that makes the diagram

$$\begin{array}{ccc} \mathcal{M}(\Omega) & \longrightarrow & \mathcal{M}(L) \\ \downarrow & & \downarrow \psi \\ \mathcal{M}(K) & \xrightarrow{\phi} & \mathcal{X} \end{array}$$

2-commutative.

Write  $|\mathcal{X}|$  for the set of equivalence classes of points of  $\mathcal{X}$ . A morphism  $f : \mathcal{X} \rightarrow \mathcal{Y}$  of  $G$ -analytic stacks naturally induces a map

$$|f| : |\mathcal{X}| \longrightarrow |\mathcal{Y}|$$

and  $|\mathcal{X}|$  is endowed with the coarsest topology making all maps  $|U| \rightarrow |\mathcal{X}|$ , that are induced by surjective and universally submersive morphisms  $U \rightarrow \mathcal{X}$  from an analytic space  $U$ , continuous.

We refer to  $|\mathcal{X}|$  as the *underlying topological space* of  $\mathcal{X}$ . Given a morphism  $f : \mathcal{X} \rightarrow \mathcal{Y}$  of  $G$ -analytic stacks, the induced morphism  $|f| : |\mathcal{X}| \rightarrow |\mathcal{Y}|$  is continuous and the association  $f \mapsto |f|$  is functorial.

**5.2. The  $(.)^\triangleright$ -functor.** From now on suppose again that  $k$  is endowed with the trivial absolute value. Using the terminology developed above we may now define a generalization of Thuillier's  $(.)^\triangleright$ -functor to algebraic stacks. Our construction proceed in two steps:

- (i) Let  $X$  be an algebraic space that is locally of finite type over  $k$  and choose a presentation  $R \rightrightarrows U$  of  $X$  by an étale equivalence relation. The induced arrows  $R^\triangleright \rightrightarrows U^\triangleright$  define a universally submersive,  $G$ -étale, surjective groupoid in  $\mathbf{An}_k$  and therefore the quotient stack  $[U^\triangleright/R^\triangleright]$  is a  $G$ -analytic space, which we denote by  $X^\triangleright$ .
- (ii) Let now  $\mathcal{X}$  be an algebraic stack that is locally of finite type over  $k$ . Choose a groupoid presentation  $[U/R] \simeq \mathcal{X}$  in the category of algebraic spaces, locally of finite type over  $k$ . The induced groupoid  $(R^\triangleright \rightrightarrows U^\triangleright)$  in the category of  $G$ -analytic spaces is universally submersive,  $G$ -smooth, and surjective. Therefore we may define  $\mathcal{X}^\triangleright$  as the  $G$ -analytic quotient stack  $[U^\triangleright/R^\triangleright]$ .

**Proposition 5.5.** *The association  $\mathcal{X} \mapsto \mathcal{X}^\triangleright$  is well-defined (up to equivalence) and defines a functor*

$$(.)^\triangleright : \mathbf{Alg.Stacks}_{\text{loc.f.t./}k} \longrightarrow \mathbf{An.Stacks}_k^G$$

*from the 2-category of algebraic stacks, locally of finite type over  $k$ , to the 2-category of  $G$ -analytic stacks.*

*Proof.* For the sake of simplicity we assume that all algebraic stacks admit a groupoid presentation in the category of schemes. The more general case is more tedious, but uses the same ideas; it is left to the avid reader.

Recall that a morphism  $f : (R \rightrightarrows U) \rightarrow (R' \rightrightarrows U')$  of algebraic (or analytic) groupoids is said to be an *equivalence*, if  $U \rightarrow U'$  is surjective and the square

$$\begin{array}{ccc} R & \longrightarrow & R' \\ (s,t) \downarrow & & \downarrow (s',t') \\ U \times U & \longrightarrow & U' \times U' \end{array}$$

is a fiber product. As proven in [Sta16, Tag 046R], an equivalence  $f : (R \rightrightarrows U) \xrightarrow{\sim} (R' \rightrightarrows U')$  of algebraic (resp. analytic) groupoids induces an equivalence  $[f] : [U/R] \xrightarrow{\sim} [U'/R']$ .

Given two groupoid presentations  $[U/R] \simeq \mathcal{X} \simeq [U'/R']$  of  $\mathcal{X}$  there is a third analytic groupoid  $(R'' \rightrightarrows U'')$  together with equivalences as indicated below:

$$\begin{array}{ccc} & (R'' \rightrightarrows U'') & \\ \sim \swarrow & & \searrow \sim \\ (R \rightrightarrows U) & & (R' \rightrightarrows U') \end{array}$$

By applying  $(.)^\triangleright$ , the induced diagram of equivalences in  $\mathbf{An}_k$  shows that there are equivalences

$$[U^\triangleright/R^\triangleright] \simeq [(U'')^\triangleright/(R'')^\triangleright] \simeq [(U')^\triangleright/(R')^\triangleright] .$$

Finally, given a (not necessarily representable) morphism  $f : [\mathcal{U}/\mathcal{R}] \rightarrow [\mathcal{U}'/\mathcal{R}']$  of algebraic stacks. Then there is a third analytic groupoid  $(\mathcal{R}'' \rightrightarrows \mathcal{U}'')$ , an equivalence  $(\mathcal{R}'' \rightrightarrows \mathcal{U}'') \xrightarrow{\sim} (\mathcal{R} \rightrightarrows \mathcal{U})$ , as well as a morphism  $g : (\mathcal{R}'' \rightrightarrows \mathcal{U}'') \rightarrow (\mathcal{R}' \rightrightarrows \mathcal{U}')$  such that the diagram

$$\begin{array}{ccc} & [\mathcal{U}''/\mathcal{R}''] & \\ \swarrow \simeq & & \searrow [g] \\ [\mathcal{U}/\mathcal{R}] & \xrightarrow{f} & [\mathcal{U}'/\mathcal{R}'] \end{array}$$

commutes. This allows us to define  $f^\triangleright$  as the composition

$$[\mathcal{U}^\triangleright/\mathcal{R}^\triangleright] \simeq [(\mathcal{U}'')^\triangleright/(\mathcal{R}'')^\triangleright] \xrightarrow{g^\triangleright} [(\mathcal{U}')^\triangleright/(\mathcal{R}')^\triangleright].$$

We leave the elementary, yet tedious, verification that this is well-defined and functorial (up to 2-isomorphism) to the avid reader and also refrain from making the operation of  $(.)^\triangleright$  on 2-morphisms explicit.  $\square$

**5.3. Structure and reduction maps.** Let  $\mathcal{X}$  be an algebraic stack that is locally of finite type over  $k$ . One can alternatively describe the underlying topological space  $|\mathcal{X}^\triangleright|$  of  $\mathcal{X}^\triangleright$  as follows:

- Its points are equivalence classes of pairs  $(R, \phi)$  consisting of a valuation ring extending  $k$  and a morphism  $\phi : \text{Spec } R \rightarrow \mathcal{X}$ .
- Two such pairs  $(R, \phi)$  and  $(S, \psi)$  are hereby *equivalent*, if there is a valuation ring  $\mathcal{O}$  extending both  $R$  and  $S$  making the diagram

$$\begin{array}{ccc} \text{Spec}(\mathcal{O}) & \longrightarrow & \text{Spec}(S) \\ \downarrow & & \downarrow \psi \\ \text{Spec}(R) & \xrightarrow{\phi} & \mathcal{X} \end{array}$$

2-commutative.

As in [Thu07, Section 1] there is a continuous *structure map*

$$\rho : |\mathcal{X}^\triangleright| \longrightarrow |\mathcal{X}|$$

as well as an anticontinuous *reduction map*

$$r : |\mathcal{X}^\triangleright| \longrightarrow |\mathcal{X}|$$

from  $|\mathcal{X}^\triangleright|$  into the space  $|\mathcal{X}|$  of Zariski points of  $\mathcal{X}$  (see [Sta16, Tag 04XE]). Let  $x = (R, \phi)$  be a point of  $\mathcal{X}^\triangleright$ . Then the image  $\rho(x)$  under the structure map is the point in  $|\mathcal{X}|$  represented by the  $\text{Spec } K \rightarrow \mathcal{X}$  induced from  $\phi$ , where  $K$  is the quotient field of  $R$ , and the image  $r(x)$  under the reduction map is represented by  $\text{Spec } \kappa \rightarrow \mathcal{X}$  induced from  $\phi$ , where  $\kappa$  is the residue field of  $R$ .

## 6. TROPICALIZATION AND ANALYTIFICATION

**6.1. Tropicalization via Kato fans – a reminder.** Let  $F$  be a Kato fan. Recall from [Uli13, Section 3] that we can associate to  $F$  its *extended cone complex*  $\bar{\Sigma}_F = F(\bar{\mathbb{R}}_{\geq 0})$  as the set of  $\bar{\mathbb{R}}_{\geq 0}$ -valued points. The extended cone complex comes with two natural maps, the continuous *structure map*  $\rho : \bar{\Sigma}_F \rightarrow F$  defined by  $u \mapsto u(\{\infty\})$  and the anticontinuous *reduction map*  $r : \bar{\Sigma}_F \rightarrow F$  defined by  $u \mapsto u(\bar{\mathbb{R}}_{>0})$ . For every open affine subset  $U = \text{Spec } P$  in  $F$  the preimage  $r^{-1}(U)$  is the canonical

compactification  $\overline{\sigma}_P = \text{Hom}(P, \overline{\mathbb{R}}_{\geq 0})$  of the rational polyhedral cone  $\text{Hom}(P, \mathbb{R}_{\geq 0})$ . Moreover, if  $F$  is irreducible with generic point  $\eta$  the preimage  $\rho^{-1}(\eta)$  is a rational polyhedral cone complex  $\Sigma_F = F(\mathbb{R}_{\geq 0})$  in the sense of [KKMSD73] that is canonically compactified by  $\overline{\Sigma}_F$ .

Suppose first that  $X$  is a logarithmic scheme that has no monodromy. By [Uli13, Proposition 4.14] there is an initial strict morphism  $\overline{\phi}_X : (X, \overline{M}_X) \rightarrow F_X$  into a Kato fan  $F_X$ , the *characteristic morphism* of  $X$  into the Kato fan  $F_X$  associated to  $X$  (see Example 3.9). Following [Uli13, Section 6.1] the tropicalization map  $\text{trop}_X : X^\square \rightarrow \Sigma_X$  into the *extended cone complex*  $\overline{\Sigma}_X = \overline{\Sigma}_{F_X} = F_X(\overline{\mathbb{R}}_{\geq 0})$  associated to  $X$  is defined as follows: A point  $x \in X^\square$  can be represented by a morphism  $\underline{x} : \text{Spec } R \rightarrow (X, \overline{O}_X)$ ; its image  $\text{trop}_X(x)$  in  $\overline{\Sigma}_X$  is given by the composition

$$\text{Spec } \overline{\mathbb{R}}_{\geq 0} \xrightarrow{\text{val}^\#} \text{Spec } R \xrightarrow{\underline{x}} (X, \overline{O}_X) \xrightarrow{\overline{\phi}_X} F_X,$$

where  $\text{val}^\#$  is the morphism of Kato fans induced by the valuation  $\text{val} : R \rightarrow \overline{\mathbb{R}}_{\geq 0}$  on  $R$ . It has been shown in [Uli13] that  $\text{trop}_X$  is well-defined, continuous, and functorial with respect to logarithmic morphisms.

**Example 6.1.** If both  $X = \text{Spec } A$  and  $F_X = \text{Spec } P$  are affine, we obtain the local tropicalization map

$$\text{trop}_X : X^\square \longrightarrow \overline{\sigma}_P,$$

introduced in [PPS13], where  $\overline{\sigma}_P$  canonically compactified cone  $\text{Hom}(P, \overline{\mathbb{R}}_{\geq 0})$ . In the special case that  $X$  is the affine toric variety  $U_P$  defined by a fine and saturated monoid  $P$  the tropicalization map

$$\text{trop}_P : U_P^\square \longrightarrow \overline{\sigma}_P$$

is given by associating to  $x \in U_P^\square$  the homomorphism  $p \mapsto -\log |x^p|_x$  and is nothing but the restriction of the Kajiwara-Payne tropicalization map (see [Kaj08] and [Pay09]) to  $U_P^\square \subseteq U_P^{\text{an}}$ .

Let  $X$  be a general logarithmic scheme. We refer to a surjective strict étale morphism  $X' \rightarrow X$ , where  $X'$  is a Zariski logarithmic scheme without monodromy, as a *simple strict étale cover* of  $X$ . As in [Uli13, Section 6.2] we define the *generalized extended cone complex*  $\overline{\Sigma}_X$  associated to  $X$  as the colimit of all extended cone complexes  $\overline{\Sigma}_{X'}$  associated to simple strict étale covers  $X' \rightarrow X$ . Since  $X^\square$  is the topological colimit of all  $(X')^\square$ , the tropicalization map  $\text{trop}_X : X^\square \rightarrow \overline{\Sigma}_X$  may therefore be constructed using the universal property of colimits.

**6.2. Analytification of Artin fans.** The purpose of this section is to prove Theorem 1.1 from the introduction. In the following Proposition 6.2 we first consider the case of an affine toric variety  $U_P = \text{Spec } k[P]$ , defined by a fine and saturated monoid  $P$ .

**Proposition 6.2.** *Let  $P$  be a fine and saturated monoid. Then there is a natural homeomorphism  $\mu_P : |\mathcal{A}_P^\square| \xrightarrow{\sim} \overline{\sigma}_P$  that makes the diagram*

$$\begin{array}{ccc} & U_P^\square & \\ \phi_X^\square \swarrow & & \searrow \text{trop}_P \\ |\mathcal{A}_P^\square| & \xrightarrow[\mu_P]{\sim} & \overline{\sigma}_P \end{array}$$

*commute.*

Proposition 6.2 is essentially a special case of [Uli14, Theorem 1.1] and its proof (provided below) is a variation of the proof in [Uli14].

We begin by recalling the construction of the non-Archimedean skeleton of the analytic space  $U_P^\triangleright$  associated to the toric variety  $U_P$ , due to Thuillier [Thu07, Section 2]. Consider the natural continuous tropicalization map

$$\begin{aligned} \text{trop}_P : U_P^\triangleright &\longrightarrow \overline{\sigma}_P = \text{Hom}(P, \overline{\mathbb{R}}_{\geq 0}) \\ x &\longmapsto \left( p \mapsto -\log |\chi^p|_x \right) \end{aligned}$$

essentially as introduced in [Kaj08] and [Pay09]. It has a natural continuous section  $J_P : \overline{\sigma}_P \longrightarrow U_P^\triangleright$  defined by associating to  $u \in \sigma_P = \text{Hom}(P, \overline{\mathbb{R}}_{\geq 0})$  the seminorm  $J_P(u)$  on  $k[P]$ , which is given by

$$J_P(u)(f) = \max_{p \in P} |a_p| \exp(-u(p))$$

for  $f = \sum_{p \in P} a_p \chi^p \in k[P]$ . The composition  $\mathbf{p}_P = J_P \circ \text{trop}_P : U_P^\triangleright \rightarrow U_P^\triangleright$  is a continuous retraction map and given by associating to  $x \in U_P^\triangleright$  the seminorm  $\mathbf{p}_P(x)$  that is determined by

$$\mathbf{p}_P(x)(f) = \max_{p \in P} |a_p| |\chi^p|_x$$

for  $f = \sum_{p \in P} a_p \chi^p \in k[P]$ . Its image  $\mathfrak{S}(U_P)$  is called the *non-Archimedean skeleton* of  $U_P^\triangleright$  and is naturally homeomorphic to  $\overline{\sigma}_P$  via  $J_P$ . Moreover, the results of [Thu07, Section 2.2] show that  $\mathbf{p}_P$  is in fact a strong deformation retraction.

Denote by  $\mu : T \times U_P \rightarrow U_P$  the operation of  $T = \text{Spec } k[M]$ , with  $M = P^{gp}$ , on  $U_P$  and note that this morphism is induced by the homomorphism

$$\begin{aligned} \mu^\# : k[P] &\longrightarrow k[M] \otimes_k k[P] \\ \chi^p &\longmapsto \chi^p \otimes \chi^p . \end{aligned}$$

Moreover consider the second projection map  $\pi : T \times U_P \rightarrow U_P$  induced by the homomorphism

$$\begin{aligned} \pi^\# : k[P] &\longrightarrow k[M] \otimes_k k[P] \\ \chi^p &\longmapsto 1 \otimes \chi^p . \end{aligned}$$

**Lemma 6.3.** *Let  $x \in U_P^\triangleright$  and consider the point  $\eta \hat{\otimes} x \in T^\triangleright \times U_P^\triangleright$  given by the seminorm*

$$|f|_{\eta \hat{\otimes} x} = \max_{m \in M} |a_m| |f_m|_x$$

*for an element  $f = \sum_{m \in M} a_m \chi^m \otimes f_m \in k[M] \otimes_k k[P]$  with unique regular functions  $f_m \in k[P]$ . Then we have*

$$\pi^\triangleright(\eta \hat{\otimes} x) = x$$

*as well as*

$$\mu^\triangleright(\eta \hat{\otimes} x) = \mathbf{p}_P(x) .$$

*Proof.* Let  $f = \sum_{p \in P} a_p \chi^p \in k[P]$ . Then we have

$$|f|_{\pi^\triangleright(\eta \hat{\otimes} x)} = \left| \sum_{p \in P} a_p 1 \otimes \chi^p \right|_{\eta \hat{\otimes} x} = |1 \otimes f|_{\eta \hat{\otimes} x} = |f|_x$$

as well as

$$|f|_{\mu^\triangleright(\eta \hat{\otimes} x)} = \left| \sum_{p \in P} a_p \chi^p \otimes \chi^p \right|_{\eta \hat{\otimes} x} = \max_{p \in P} |a_p| |\chi^p|_x = |f|_{\mathbf{p}_P(x)}$$

and this implies the claim. □



*Proof of Proposition 6.2.* The topological space  $|\mathcal{U}_P^\triangleright/T^\triangleright|$  is the topological colimit of the two maps

$$(\pi^\triangleright, \mu^\triangleright : T^\triangleright \times \mathcal{U}_P^\triangleright \rightrightarrows \mathcal{U}_P^\triangleright). \quad (1)$$

Therefore it is enough to show that the deformation retraction  $\mathbf{p}_P : \mathcal{U}_P^\triangleright \rightarrow \mathfrak{S}(\mathcal{U}_P)$  makes the skeleton  $\mathfrak{S}(\mathcal{U}_P)$  into a topological colimit of the above two maps.

- Let  $x, x' \in \mathcal{U}_P^\triangleright$  and  $y \in T^\triangleright \times \mathcal{U}_P^\triangleright$  with  $\pi^\triangleright(y) = x$  and  $\mu^\triangleright(y) = x'$ . In this case we have  $\mathbf{p}_P(x) = \mathbf{p}_P(x')$ , since

$$\begin{aligned} |\chi^p|_{x'} &= |\chi^p|_{\mu^\triangleright(y)} = |\chi^p \otimes \chi^p|_y \\ &= |\chi^p \otimes 1|_y \cdot |1 \otimes \chi^p|_y = |1 \otimes \chi^p|_y \\ &= |\chi^p|_{\pi^\triangleright(y)} = |\chi^p|_x \end{aligned}$$

for all  $p \in P$  using the fact that  $|\chi^m \otimes 1|_y = 1$  for all  $m \in M$ .

- For  $x \in \mathcal{U}_P^\triangleright$  Lemma 6.3 implies that there is a point  $y = \eta \hat{\otimes} x \in T^\triangleright \otimes \mathcal{U}_P^\triangleright$  such that  $\pi^\triangleright(y) = x$  and  $\mu^\triangleright(y) = \mathbf{p}_P(x)$ . So, given two points  $x, x' \in \mathcal{U}_P^\triangleright$  such that  $\mathbf{p}_P(x) = \mathbf{p}_P(x')$ , their image in  $|\mathcal{U}_P^\triangleright/T^\triangleright|$  is equal as well.

That is why the skeleton  $\mathfrak{S}(\mathcal{U}_P)$  is the set-theoretic colimit of (1) and, since  $\mathbf{p}_P$  is continuous and proper, this is actually a colimit in the category of topological spaces.  $\square$

**Corollary 6.4.** *Let  $F$  be a Kato fan. Then there is a natural homeomorphism  $\mu_F : |\mathcal{A}_F^\triangleright| \xrightarrow{\sim} \overline{\Sigma}_F$  that makes the diagrams*

$$\begin{array}{ccc} |\mathcal{A}_F^\triangleright| & \xrightarrow{\mu_F} & \overline{\Sigma}_F \\ r \downarrow & & \downarrow r \\ (|\mathcal{A}_F|, \overline{M}_{\mathcal{A}_F}) & \xrightarrow{\simeq} & F \end{array} \quad \begin{array}{ccc} |\mathcal{A}_F^\triangleright| & \xrightarrow{\mu_F} & \overline{\Sigma}_F \\ \rho \downarrow & & \downarrow \rho \\ (|\mathcal{A}_F|, \overline{M}_{\mathcal{A}_F}) & \xrightarrow{\simeq} & F \end{array}$$

commute.

*Proof.* Let  $P \rightarrow Q$  be a homomorphism between fine and saturated monoids inducing an open immersion  $\text{Spec } Q \hookrightarrow \text{Spec } P$  of affine Kato fans. Since the diagram

$$\begin{array}{ccc} \mathfrak{S}(\mathcal{U}_Q) & \xleftarrow[\sim]{J_Q} & \overline{\sigma}_Q \\ \subseteq \downarrow & & \downarrow \subseteq \\ \mathfrak{S}(\mathcal{U}_P) & \xleftarrow[\sim]{J_P} & \overline{\sigma}_P \end{array}$$

naturally commutes, the homeomorphisms from Proposition 6.2 make the diagram

$$\begin{array}{ccc} |\mathcal{A}_Q^\triangleright| & \xrightarrow[\sim]{\mu_Q} & \overline{\sigma}_Q \\ \subseteq \downarrow & & \downarrow \subseteq \\ |\mathcal{A}_P^\triangleright| & \xrightarrow[\sim]{\mu_P} & \overline{\sigma}_P \end{array}$$

commute as well. Therefore we can glue the  $\mu_P$  on open affine patches and obtain a global homeomorphism  $\mu_F : |\mathcal{A}_F^\triangleright| \xrightarrow{\sim} \overline{\Sigma}_F$ . The commutativity with respect to reduction and structure maps now immediately follows from [Uli13, Proposition 6.2 (i)].  $\square$

**Corollary 6.5.** *Let  $X$  be a Zariski logarithmic scheme without monodromy. Then there is a natural homeomorphism  $\mu_X : |\mathcal{A}_X^\triangleright| \xrightarrow{\sim} \overline{\Sigma}_X$  that makes the diagram*

$$\begin{array}{ccc}
& X^\triangleright & \\
\phi_X^\triangleright \swarrow & & \searrow \text{trop}_X \\
|\mathcal{A}_X^\triangleright| & \xrightarrow[\mu_X]{\sim} & \bar{\Sigma}_X
\end{array}$$

commute.

*Proof.* The homeomorphism  $\mu_X = \mu_{F_X}$  has already been constructed in the above Corollary 6.4. In order to check the commutativity of the above diagram we may assume that  $X$  is small. In this case  $P = P_X$  lifts to a chart of the logarithmic structure on  $X$ , we have  $\bar{\Sigma}_X = \bar{\sigma}_P$ , and the tropicalization map  $\text{trop}_X$  factors as

$$X^\triangleright \longrightarrow U_P^\triangleright \xrightarrow{\text{trop}_P} \bar{\sigma}_P.$$

Therefore the above diagram is commutative by Proposition 6.2.  $\square$

*Proof of Theorem 1.1.* Let  $X$  be a logarithmic scheme. For every simple strict étale cover  $X' \rightarrow X$  of  $X$  by a Zariski logarithmic scheme without monodromy we obtain a homeomorphism  $\mu_{X'} : |\mathcal{A}_{X'}^\triangleright| \xrightarrow{\sim} \bar{\Sigma}_{X'}$  and the  $\mu_{X'}$  are compatible with simple strict étale covers. Therefore, since  $X^\triangleright$  is the topological colimit of the  $(X')^\triangleright$  by [ACP15, Lemma 6.1.3], we only need to show that  $|\mathcal{A}_X^\triangleright|$  is the topological colimit of the  $|\mathcal{A}_{X'}^\triangleright|$ .

Consider a point  $x$  in  $|\mathcal{A}_X^\triangleright|$  represented by a morphism  $\text{Spec } R \rightarrow \mathcal{X}$  for a valuation ring  $R$  extending  $k$ , whose quotient field is separably closed. Since  $\text{Spec } R$  does not have any non-trivial étale covers, the morphism  $\text{Spec } R \rightarrow \mathcal{A}_X$  lifts to a morphism  $\text{Spec } R \rightarrow \mathcal{A}_{X'}$  for every simple strict étale cover  $X' \rightarrow X$ . Therefore  $|\mathcal{A}_{X'}^\triangleright| \rightarrow |\mathcal{A}_X^\triangleright|$  is surjective for all  $X'$ . Given two points  $x_1, x_2$  in  $|\mathcal{A}_X^\triangleright|$  whose images in  $|\mathcal{A}_X^\triangleright|$  are equal. We may again represent these points by morphisms  $\underline{x}_i : \text{Spec } R \rightarrow \mathcal{A}_{X'}$ , where  $R$  is a valuation ring extending  $k$  that has a separably closed quotient field, such that the compositions of  $\underline{x}_i$  with  $\mathcal{A}_{X'} \rightarrow \mathcal{A}_X$  are 2-isomorphic. Again, since  $\text{Spec } R$  has no non-trivial étale covers and  $\mathcal{A}_X$  is the colimit of all  $\mathcal{A}_{X'}$  in the category of sheaves over  $\text{LOG}_k$ , this is only the case, if the  $\underline{x}_i$  lift to 2-isomorphic morphisms  $\text{Spec } R \rightarrow \mathcal{A}_{X''}$  for some simple strict étale cover  $X''$  of  $X'$ .  $\square$

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